Nonlinear dynamics of wave packets in parity-time-symmetric optical lattices near the phase transition point

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Nonlinear dynamics of wave packets in parity-time-symmetric optical lattices near the phase-transition point is analytically studied. A nonlinear Klein–Gordon equation is derived for the envelope of these wave packets. A variety of phenomena known to exist in this envelope equation are shown to also exist in the full equation, including wave blowup, periodic bound states, and solitary wave solutions.

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Linear Schrödinger operators with complex but parity-time ($\mathcal{PT}$)-symmetric potentials have the unintuitive property that their spectra can be completely real [1]. This phenomenon was first studied in quantum mechanics where a real spectrum is required to guarantee real energy levels. The same phenomenon was later investigated in optics, where $\mathcal{PT}$-symmetric potentials could be realized by employing symmetric index guiding and an antisymmetric gain/loss profile [2–4]. In this optical setting, $\mathcal{PT}$ potentials have been experimentally realized [5,6]. In temporal optics, $\mathcal{PT}$-symmetric lattices have been experimentally obtained as well [7]. So far, a number of physical phenomena in optical $\mathcal{PT}$ systems have been reported, including phase transition ($\mathcal{PT}$-symmetry breaking), nonreciprocal Bloch oscillation, unidirectional propagation, distinct pattern of diffraction, formation of soliton families, and so on [5–12]. The search of additional new behaviors in optical $\mathcal{PT}$ systems is still ongoing.

In this Letter, we analytically study nonlinear dynamics of wave packets in $\mathcal{PT}$-symmetric optical lattices near the phase-transition point (where bandgaps close and Bloch bands intersect with each other transversely). A nonlinear Klein–Gordon equation is derived for the envelope of these wave packets near the band intersection. Based on this envelope equation, we predict a variety of phenomena, such as wave splitting, wave blowup, periodic bound states, and solitary wave states. We further show these predicted phenomena occur in the full model as well.

The paraxial model for nonlinear propagation of light beams in $\mathcal{PT}$-symmetric optical lattices is

$$i\Psi_z + \Psi_{xx} + V(x)\Psi + \sigma|\Psi|^2\Psi = 0, \tag{1}$$

where z is the propagation axis, x is the transverse axis,

$$V(x) = V_0^2[\cos(2x) + iW_0 \sin(2x)] \tag{2}$$

is a $\mathcal{PT}$-symmetric potential, V_0^2 is the potential depth, W_0 is the relative gain/loss strength, and $\sigma = \pm 1$ is the sign of nonlinearity. All variables are nondimensionalized.

First we discuss the linear diffraction relation of Eq. (1) at the phase-transition point $W_0 = 1$ [8,12]. In this case, the linear equation of (1) can be solved exactly [12]. Its Bloch solutions are

$$\Psi_\pm(x, z; \mu) = I_{\pm(k+2m)}(V_0 e^{ix}) e^{-i\mu z}, \tag{3}$$

where I_k is the modified Bessel function, $\mu = (k + 2m)^2$ is the diffraction relation, k is in the first Brillouin zone $k \in [-1, 1]$, and m is any non-negative integer. This diffraction relation is depicted in Fig. 1(A), where different colors indicate different Bloch bands. At $k = 0$ and ± 1, adjacent Bloch bands intersect each other transversely like. Because of this intersection (degeneracy), wave packets near these points will exhibit distinctive dynamics that we will reveal next.

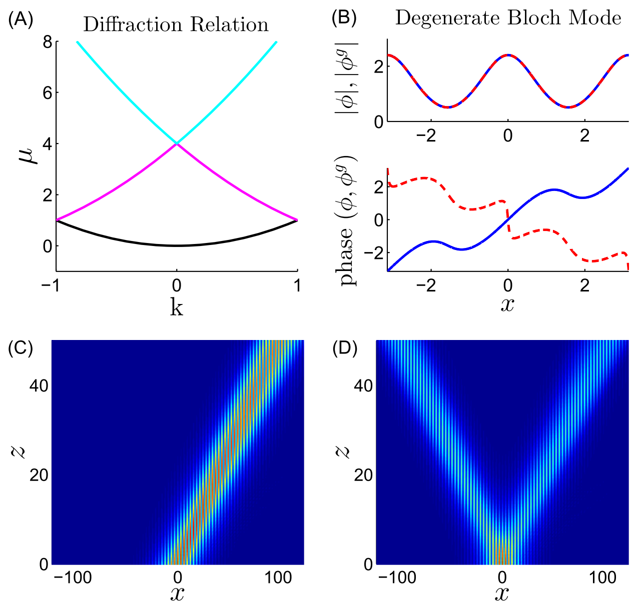

Fig. 1. (Color online) (A) Diffraction relation. (B) Magnitude and phase of eigenfunction ϕ (solid blue) and generalized eigenfunction ϕ^g (dashed red) for $\mu = 1$ and $V_0 = \sqrt{6}$. (C) Linear unidirectional wavepacket and (D) linear wavepacket splitting in Eq. (1) at the phase-transition point $W_0 = 1$.

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At these intersection points, $\Psi_+ = \Psi_-$. Thus Bloch solutions (3) are degenerate and 2π periodic in x. Posed as an eigenvalue problem for $\Psi = \phi(x)e^{-i\mu z}$ in the linear equation of (1), we get $L\phi = -\mu\phi$, where $L \equiv \partial_{xx} + V_0(x)$, and $V_0(x)$ is the $\mathcal{PT}$ lattice at the phase-transition point $W_0 = 1$. Then at these band-intersection points, the eigenvalues are $\mu = n^2$, where n is any positive integer. The corresponding eigenfunctions are

$$\phi(x) = I_n(V_0 e^{ix}) = \sum_{j=0}^{\infty} \frac{(V_0 e^{ix}/2)^{2j+n}}{j!(j+n)!}. \tag{4}$$

These eigenvalues $\mu = n^2$ all have geometric multiplicity 1 and algebraic multiplicity 2; thus there exists a generalized eigenfunction ϕ^g satisfying

$$(L + \mu)\phi^g = \phi. \tag{5}$$

For $V_0 = \sqrt{6}$ and $n = 1$, this eigenfunction and generalized eigenfunction are plotted in Fig. 1(B).

Now we consider nonlinear dynamics of wave packets near these band-intersection points. For convenience, we first take the $\mathcal{PT}$ lattice to be exactly at the phase-transition point (i.e., $W_0 = 1$). Generalization to lattices near the phase-transition point will be made afterward.

Nonlinear wave-packet solutions near a band-intersection point can be expanded into a perturbation series

$$\Psi = e^{-i\mu z}[\epsilon A(X, Z)\phi(x) + \epsilon^2\psi_1 + \epsilon^3\psi_2 + \ldots], \tag{6}$$

where $\mu = n^2$ is the propagation constant at the band intersection, $\phi(x)$ is the degenerate Bloch mode given in Eq. (4), $A(X, Z)$ is the envelope of this Bloch mode, $X = \epsilon x$, $Z = \epsilon z$ are slow variables, and $0 < \epsilon \ll 1$ is a small positive parameter. Substituting the above perturbation series into the original Eq. (1), this equation at $O(\epsilon)$ is automatically satisfied. At order ϵ^2 we have

$$(L + \mu)\psi_1 = -iA_Z\phi - 2A_X\phi_x. \tag{7}$$

The solution to this equation is

$$\psi_1 = -iA_Z\phi^g - 2A_X\phi^d, \tag{8}$$

where ϕ^g is the generalized Bloch mode defined in (5), and

$$\phi^d = (L + \mu)^{-1}\phi_x. \tag{9}$$

This ϕ^d solution exists and can be determined by Fourier series [12].

At $O(\epsilon^3)$ we get

$$\begin{aligned}(L + \mu)\psi_2 = {} & -A_{ZZ}\phi^g + A_{XX}(4\phi^d_x - \phi) \\ & + i2A_{ZX}(\phi^d + \phi^g_x) - \sigma|A|^2A|\phi|^2\phi. \end{aligned}\tag{10}$$

The solvability condition of this equation is that its right-hand side be orthogonal to the adjoint homogeneous solution ϕ^*. Using Fourier expansions of solutions ϕ^g and ϕ^d (see [12]), this solvability condition then yields the following nonlinear Klein–Gordon equation for the envelope function $A(X, Z)$:

$$A_{ZZ} - 4n^2A_{XX} + \gamma|A|^2A = 0, \tag{11}$$

where

$$\gamma = (-1)^{n+1}\frac{2\sigma n^2}{\pi}\int_{-\pi}^{\pi}|\phi|^2\phi^2\mathrm{d}x. \tag{12}$$

In view of the formula (4) for ϕ, it is easy to see that γ is real and $\mathrm{sgn}(\gamma) = (-1)^{n+1}\sigma$. For the values of $V_0 = \sqrt{6}$ and $n = 1$, which we will use in later numerical simulations, $\gamma \approx 11.0430\sigma$.

Now we extend the above envelope equation to the case where the $\mathcal{PT}$ lattice is near the phase-transition point (i.e., $W_0 \sim 1$). Following similar perturbation analysis, we find that when $\mu = n = 1$ (the lowest band-intersection point) and $W_0 = 1 - c\epsilon^2$, the envelope is governed by a slightly more general nonlinear Klein–Gordon equation,

$$A_{ZZ} - 4n^2A_{XX} + \alpha A + \gamma|A|^2A = 0, \tag{13}$$

where γ is as given in (12), $\alpha = cV_0^4/2$, and the ψ_1 solution in (6) is still given by Eq. (8). When $\mu = n^2$ with $n > 1$ and $W_0 = 1 - c\epsilon$, the envelope Eq. (13) will contain an additional term proportional to iA_Z. But this iA_Z term can be eliminated through a gauge transformation $A \to Ae^{-icV_0^4/4(n^2-1)}$, and the transformed equation remains the same as (13), except that $\alpha = c^2V_0^8/64$ for $n = 2$ and $\alpha = 0$ for $n > 2$. If $c = 0$ (i.e., at the phase-transition point), then $\alpha = 0$; hence Eq. (13) reproduces Eq. (11) as a special case. When $n > 1$, the ψ_1 solution (8) will also contain an additional term proportional to icA.

The nonlinear Klein–Gordon Eq. (13) is second-order in Z, which means that two initial conditions, $A(X, 0)$ and $A_Z(X, 0)$, are needed. These two initial conditions can be obtained from the initial conditions of the perturbation series (6) at orders ϵ and ϵ^2, i.e., from the initial envelope $A(X, 0)$ as well as $\psi_1|_{z=0}$. This $\psi_1|_{z=0}$ generally contains many eigenmodes of the operator L, but only the ϕ^g component in it affects the dynamics of envelope A. If we denote $B(X)$ as the envelope function of the ϕ^g component in $\psi_1|_{z=0}$, then by projecting $\psi_1|_{z=0}$ [such as from (8)] onto ϕ^g, we find that

$$B(X) = -i[A_Z(X, 0) + 2nA_X(X, 0)]. \tag{14}$$

This relation holds for all n values. Thus, initial conditions for the original $\mathcal{PT}$ model (1) and those for the envelope Eq. (13) can be related as

$$\Psi(x, 0) = \epsilon A(X, 0)\phi(x) + \epsilon^2B(X)\phi^g(x). \tag{15}$$

This initial-value connection will be used in our numerical simulations later.

Now we examine the envelope dynamics in the nonlinear Klein–Gordon Eq. (13), and show that the corresponding wavepacket dynamics occurs in the original

$\mathcal{PT}$ model (1) too. In this discussion, we take $n = 1$, $V_0 = \sqrt{6}$ and $\epsilon = 0.1$ for definiteness.

First, we consider the linear Klein–Gordon Eq. (13) at the phase-transition point, i.e., $A_{ZZ} - 4n^2 A_{XX} = 0$. This is the familiar second-order wave equation. It admits unidirectional wave solutions $F(X - 2nZ)$ as well as bidirectional wave solutions $F(X - 2nZ) + G(X + 2nZ)$. In the original linear $\mathcal{PT}$ model (1), we find that the corresponding wave-packet solutions also exist. Examples are displayed in Figs. 1(C) and 1(D). Note that the wave splitting in Fig. 1(D) is a manifestation of the transversely-intersecting diffraction structure at the band-intersection point, while the unidirectional propagation in Fig. 1(C) occurs for $B(X) = 0$.

Next we consider envelope solutions in the nonlinear Klein–Gordon Eq. (13) below the phase-transition point, i.e., $W_0 < 1$, or $\alpha > 0$ (above the phase-transition point, infinitesimal linear waves are unstable; thus it is not pursued). When the nonlinearity is self-defocusing ($\sigma = -1$), $\gamma \approx -11.0430 < 0$. In this case, envelope solutions can blow up to infinity in finite distance [13]. One such example is shown in Fig. 2 (upper left panel) for $c = 1$ ($\alpha = 18$) and initial conditions $A(X,0) = 1.2\,\mathrm{sech}(X)$, $A_Z(X,0) = -2A_X(X,0)$ ($B(X) = 0$). In the full nonlinear $\mathcal{PT}$ model (1), we have found similar blowup solutions, which are displayed in Fig. 2 (upper right panel). This solution-blowup under self-defocusing nonlinearity is very surprising. Note that in the full model (1), our asymptotic envelope approximation breaks down as the singular (blowup) point is approached. In this case, the amplitude of the full-model solution eventually saturates, but its power still grows unbounded [12].

Under self-defocusing nonlinearity, the envelope Eq. (13) also admits X-localized and Z-periodic bound states. One example with $c = 1$ (below the phase-transition point) is shown in Fig. 2 (lower left panel). The initial condition for this solution is $A(X,0) = \mathrm{sech}(X)$ and $A_Z(X,0) = 0$ ($B(X) = -2iA_X(X,0)$). In the full nonlinear $\mathcal{PT}$ model (1), we have found similar periodic bound states as well, [see Fig. 2 (lower right panel)].

Under self-defocusing nonlinearity and below the phase-transition point, the envelope Eq. (13) also admits stationary solitary waves $A(X,Z) = F(X)e^{i\omega Z}$ when ω lies inside the bandgap $-\sqrt{\alpha} < \omega < \sqrt{\alpha}$. These solitary envelope solutions correspond to the solitons of the full nonlinear $\mathcal{PT}$ model (1) reported in [8,12].

If the nonlinearity is self-focusing ($\sigma = 1$), envelope solutions do not blow up, periodic bound states cannot be found, and stationary solitary waves do not exist in the envelope equation. In this case, nonlinear diffracting solutions similar to the linear diffracting pattern reported in [10] exist. In addition, solutions periodic in both X and Z can be found.

In summary, we have shown that the nonlinear Klein–Gordon equation governs the envelope dynamics of wavepackets in $\mathcal{PT}$ lattices near the phase-transition point. We have also shown that a variety of phenomena in this envelope equation (such as wave blowup and periodic bound states) occur in the full $\mathcal{PT}$ model too. These findings open new possibilities for wave engineering in $\mathcal{PT}$ lattices.

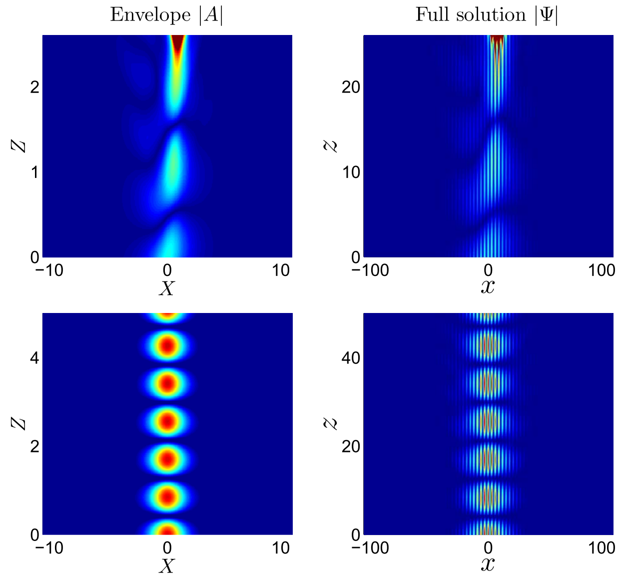

Fig. 2. (Color online) Nonlinear wavepacket solutions below the phase-transition point under self-defocusing nonlinearity. Top: a blowup solution. Bottom: a periodic bound state. Left: envelope solutions $|A|$. Right: full solutions $|\Psi|$.

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