Dark solitons in mode-locked lasers

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Dark soliton formation in mode-locked lasers is investigated by means of a power-energy saturation model which incorporates gain and filtering saturated with energy, and loss saturated with power. It is found that general initial conditions evolve (mode-lock) into dark solitons under appropriate requirements also met in experimental observations. The resulting pulses are essentially dark solitons of the unperturbed nonlinear Schrödinger equation. Notably, the same framework also describes bright pulses in anomalous and normally dispersive lasers. © 2011 Optical Society of America

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Laser frequency stabilization via mode-locked (ML) lasers has become an indispensable tool in many research activities. Advances in optical frequency standards have resulted in the development of precise frequency measurement capability in the visible and near-infrared spectral regions. Although the potential for using ML lasers in optical frequency synthesis was recognized early, the available lasers did not provide the properties necessary for fulfilling this potential until recently [1]. The recent explosion of measurements based on ML lasers has been largely due to the development of the Kerr-lens mode-locked Ti:Sapphire laser and its capability to generate sufficiently short pulses. Alternatively, amplified pulses of a similar duration are created through the compression of pulses via self-phase modulation in a hollow core fiber [2].

Dark solitons are intensity dips on a constant background with a phase jump across their intensity minimum. Since their discovery [3], dark solitons have attracted considerable attention, especially in the fields of nonlinear optics [4] and Bose-Einstein condensates [5]. In addition, the development of dark pulse laser open the door to new applications in the fields of optical communications [4] and femtosecond lasers [6]. In the non-mode-locking regime, the first train of dark solitons was successfully achieved in an all normal dispersion Erbium-doped fiber laser with an in-cavity polarizer. In ML lasers they have been difficult to generate with the first experimental observation only recently reported [2, 7]. Apart from the bright pulse emission, it was observed that, under strong continuous wave emission, appropriate pump strength and negative cavity feedback, a fiber laser can also emit single or multiple dark pulses [2].

To analyze dark solitons in ML lasers we use a model based on a variant of the nonlinear Schrödinger (NLS) equation, termed the power-energy saturation (PES) equation [8], suitably generalized to include non-vanishing boundary conditions at infinity. It is expressed in the following dimensionless form,

$$i\psi_z - \frac{1}{2}\psi_{tt} + |\psi|^2\psi =$$

$$\frac{ig}{1 + E/E_0}\psi + \frac{i\tau}{1 + E/E_0}\psi_{tt} - \frac{il}{1 + P/P_0}\psi, \quad (1)$$

where the complex electric field envelope $\psi(z,t)$ is subject to the boundary conditions $|\psi(z,t)| \to |\psi_\infty|$ as $|t| \to \infty$. Here, $E(z) = \int_{-\infty}^{+\infty}(|\psi_\infty|^2 - |\psi|^2)\,dt$ is the dark-pulse energy, $P(z,t) = |\psi_\infty|^2 - |\psi|^2$ is the instantaneous power, while E_0 and P_0 are related to the saturation energy and power, respectively. Furthermore, g, τ, and l are all positive, real constants, with the corresponding terms representing saturable gain, spectral filtering, and saturable loss. For bright solitons this type of rational function model has held up remarkably well when compared to experimental results [9]. Typical dimensional numbers can be found in Ref. [10]. Variants of Eq. (1) have been considered in the past for both bright [11] and dark [12] pulses.

The above definition of the pulse energy, although unconventional, corresponds to the physical properties of the system. Indeed, as the peak-to-background intensity ratio can achieve large values in bright pulses, so can the focused-to-unfocused vacuum ratio in dark pulses. As such, dark pulses can be thought of as focusing the vacuum [13]. The definition of power follows consistently from $E = \int_{-\infty}^{+\infty} P\,dt$. A modified model (yielding similar results), with more standard definitions of energy and power, is also discussed below.

To study the dynamics and mode-locking capabilities of our model, we integrate Eq. (1) with a fourth-order Runge-Kutta method, with initial profile $\psi(0,t) = \tanh t$, though others, with a π-phase jump, can be used as well. The gain parameter g is varied, while $E_0 = P_0 = 1$ and $\tau = l = 0.1$. In the top panel of Fig. 1 we show the evolution of the background amplitude, which sets the dark soliton amplitude, for different values of the gain parameter g. Locking onto stable dark solitons is only achieved when the gain term is sufficiently
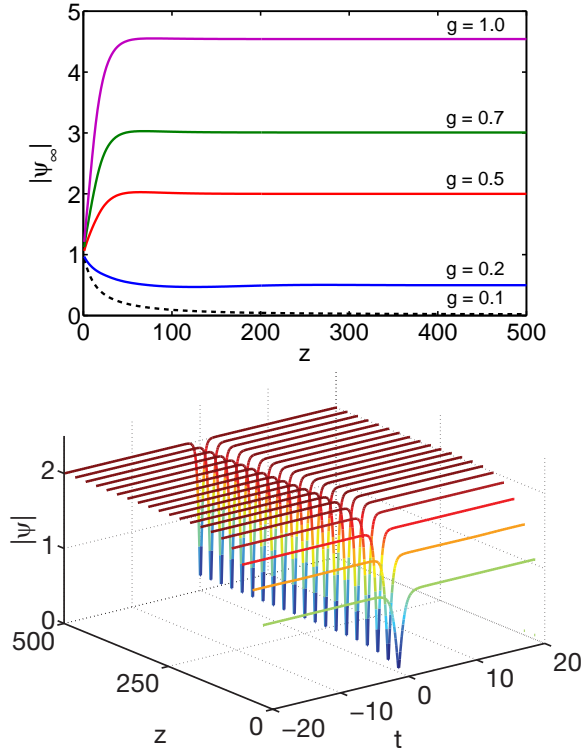
Fig. 1. (Color online) Top panel: evolution of the pulse background of an arbitrary initial profile under the PES equation with different values of gain. Bottom panel: complete evolution of the initial profile for $g = 0.5$.

strong, i.e. the parameter g is large enough to counter balance the losses. Thus, for $g = 0.1$ (see dashed line in the top panel of Fig. 1), the dark soliton decays quickly due to excessive loss. On the other hand, for $g = 0.2$, the pulse initially decreases, but then locks to a specific amplitude and width. Finally, for $g = 0.5, 0.7, 1.0$, a stable evolution characterized by an increase of the soliton amplitude is obtained almost instantaneously; see bottom panel of Fig. 1.

To determine the evolution of the background wave in the framework of Eq. (1), we assume that $\psi(z) = \psi_\infty(z) \exp[i\theta(z)]$. Separating real and imaginary parts, yields for the background amplitude $\psi_\infty(z)$:

$$\frac{\mathrm{d}\psi_\infty}{\mathrm{d}z} = \frac{g}{1 + 2|\psi_\infty|/E_0}\psi_\infty - l\psi_\infty. \qquad (2)$$

Here, an approximate solution of Eq. (1) in the form $|\psi(z,t)| = |\psi_\infty \tanh(\psi_\infty t)|$ is assumed, which gives $E = 2|\psi_\infty|$. Equation (2) also describes the evolution of ψ_∞, as depicted in the top panel of Fig. 1, and illustrates the importance of energy saturation. Indeed, if we consider the so-called similariton supporting equation (NLS with linear gain) corresponding to $\tau = l = 0$ [10, 14], with energy saturation absent ($2|\psi_\infty|/E_0 = 0$), the pulse amplitude grows exponentially at a rate defined by the gain g. Contrary, when saturation is present, this exponentially growing state cannot be reached. In some models,

loss is introduced in the form of fast saturable power absorbers placed periodically [15]. It has been shown that the lumped and distributive models yield similar results [16].

A stable equilibrium (attractor) exists for Eq. (2) and can be found setting $\mathrm{d}\psi_\infty/\mathrm{d}z = 0$, namely,

$$|\psi_\infty| = \frac{E_0}{2}\left(\frac{g}{l} - 1\right). \qquad (3)$$

This is the resulting background amplitude of the dark soliton and agrees with direct numerical simulation, as seen in Fig. 2. Thus, dark solitons tend to an equilibrium (mode-lock) with constant energy and background.

The above findings agree with the experiment of Ref. [2], where the mode-locked dark pulses were identified as NLS dark solitons. In particular, they are *black* solitons (stationary kinks) characterized by a π-phase jump across the soliton notch and they can be described analytically by a hyperbolic tangent profile, i.e., $\psi(z,t) = \psi_\infty \tanh(\psi_\infty t)$. In Fig. 2 we plot the black solitons of the PES and NLS equations (for the same value of ψ_∞, as determined from Eq. (3)) for different values of g. The amplitudes match so closely that they are in-
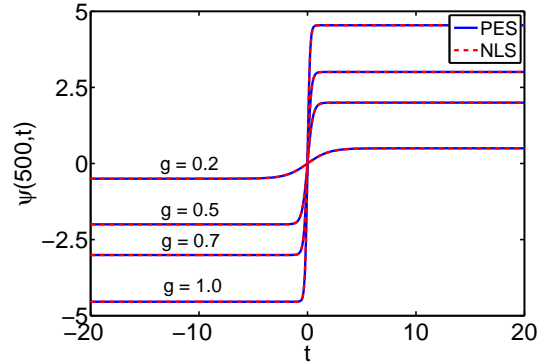

Fig. 2. (Color online) Black solitons of the PES and NLS equations for different values of g.

distinguishable in the figure, meaning that the combined perturbation of gain, filtering and loss in the PES model underlies the mode-locking mechanism: its effect is to mode-lock to a black soliton of the pure NLS equation with the appropriate background, Eq. (3). On the other hand, mode-locking to *grey* solitons, namely moving dark pulses with a phase-jump less than π at their (non-zero) intensity minima, are not found in this model as they have insufficient dark-pulse energy. This too agrees with Ref. [2] where grey solitons were not observed. Note that recent experimental [17] and theoretical [18] results in ML lasers in the anomalous dispersion regime indicate that the normalized intensity of a pulse in a ML laser can be described by the bright solitons of the unperturbed NLS equation.

The conventional definition of the pulse energy may also be used in Eq. (1), namely $E = \frac{1}{T}\int_{-T/2}^{T/2}|\psi|^2\,\mathrm{d}t$, where T is the averaged time through the cavity. In this

case, looking for stationary solutions of Eq. (3) yields

$$|\psi_\infty|^2 = \frac{g - l}{l/E_0 - g/P_0} \qquad (4)$$

for the background where, for sufficiently large T, $E = \frac{1}{T}\int_{-T/2}^{T/2}|\psi|^2\,\mathrm{d}t \approx \frac{1}{T}|\psi_\infty|^2\,T = |\psi_\infty|^2$. Equation (4) is a more restrictive condition for mode-locking which can also be attributed to the experimental difficulties of the problem. Not only must the gain be greater than the loss as in Eq. (3), but the gain must saturate faster with respect to increases in background intensity. This definition also results in more complicated dynamics and the development of a *shelf* on the soliton background, which is more pronounced as compared to the previous case. In fact, as shown elsewhere [19], shelves occur naturally in the perturbation theory for dark solitons. We illustrate the resulting shelf in Fig. 3, where now $g = 0.2$, $l = 0.1$, $E_0 = 1$ and $P_0 = 5$. Due to the noisy background
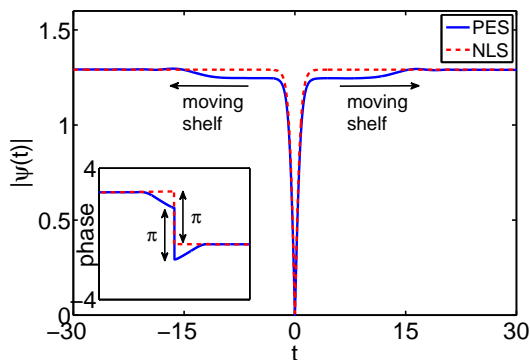

Fig. 3. (Color online) The development of a shelf in the solution to the PES equation (solid line) as compared to the solution of the NLS equation (dashed line).

this feature maybe difficult to observe in an experiment. However, the shelf also affects the phase of the resulting pulse (see inset in Fig. 3) and since the experimentally observed pulses exhibit a sharp π-phase jump [2], the first dark-pulse energy definition may give a closer approximation for this problem.

Lastly, we mention the notion of multiple dark solitons. To observe these, we evolve Eq. (1) with initial condition $\psi(0,t) = \tilde{\psi}_\infty \tanh(\tilde{\psi}_\infty t + t_0)$ for $t < 0$ and $\psi(0,t) = -\tilde{\psi}_\infty \tanh(\tilde{\psi}_\infty t - t_0)$ for $t > 0$, where, $\tilde{\psi}_\infty$ can be approximated by $\psi_\infty/2$; the dark-pulse energy is now twice that of the single soliton. In Fig. 4 we show its evolution. As in the case of bright pulses [20], the two solitons are taken initially sufficiently far apart to minimize interactions. The major difference with the bright case, where the interaction is logarithmically slow [20], is that pulses repel sooner here due to the shelf interactions. While the solitons are stable for reasonably long distance enough to also be experimentally observed, nevertheless one can expect this case to be sensitive as with the single dark soliton.
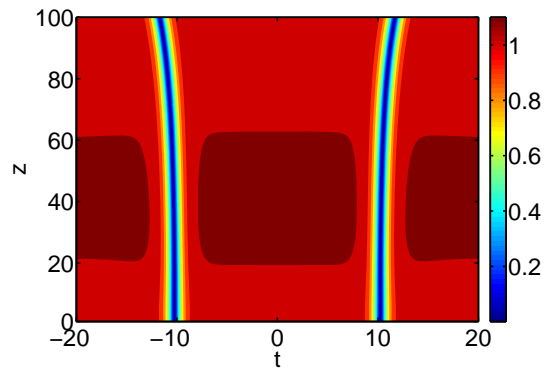

Fig. 4. (Color online) Two-dark-soliton evolution.

In conclusion, we presented a power-energy saturation model that describes mode-locking of dark solitons in lasers. Much like their bright counterparts, these pulses (as well as multiple pulses) exist when sufficient gain is present in the system, which agrees with the experimental observations. The specific energy saturated gain and filtering, and power saturated loss, are crucial to the mode-locking mechanism. The resulting pulses are essentially modes of the unperturbed NLS equation with background amplitudes appropriately defined by the gain and loss parameters.

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